Fixed Point Results for α-Admissible Mappings in Rectangular Metric Spaces

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Abstract: In this paper, we shall prove the fixed point theorems in rectangular metric space for generalized contractions using α-admissible mappings. In the end, we shall discuss about consequences of our main results.

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1. **Introduction:** In 1922, Banach gave a principle to obtain the fixed point in the complete metric space. Since then, many researchers have worked on the Banach fixed point theorem (see [1-9], [11-22]) and tried to generalize this principle. In 2012, Samet *et al.* [23] introduced the new concepts of mappings called α-admissible mappings in metric space. Recently, in 2013 Farhan *et al.* [2] gave new contractions using α-admissible mapping in metric spaces.

   In this paper, we shall generalize Farhan's *et al.* [2] contractions and give fixed point theorems for such contractions.

2. **Preliminaries:** To prove our main results we need some basic definitions from literature as follows:

Definition 2.1. [10] Let X be a set. A rectangular metric space (RMS) is an ordered pair (X, d) where d is a function $d : X \times X \rightarrow \mathbb{R}$ such that

(1) $(x, y) \geq 0$,

(2) $(x, y) = 0$ iff $x = y$,

(3) $(x, y) = d(y, x)$,

(4) $(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

For all $x, y, u, v \in$.

Definition 2.2. [10] A sequence $\{x_n\}$ in RMS (X, d) is said to converge if there is a point $x \in X$ and for every $\in > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \in$ for every $n > N$.

Definition 2.3. [10] A sequence $\{x_n\}$ in a RMS (X, d) is Cauchy if for every $\in > 0$ there exists $N \in \mathbb{N}$ such that $(x_n, x_m) < \in$ for every $n, m > N$.

Definition 2.4. [10] RMS (X, d) is said to be complete if every Cauchy sequence is convergent.

Definition 2.5. [23] Let $f: X \to X$ and $\alpha: X \times X \to [0, \infty)$. We say that f is an α −admissible mapping if

$(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1,\ x, y \in X$.

3. Main Results:

Theorem 3.1. Let (X, d) be a complete RMS and $T: X \to X$ be an α − admissible mapping. Assume that there exists a function $\beta: [0, \infty) \to [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $(t_n) \to 1$ implies $t_n \to 0$ and

$$(d(Tx, Ty) + l)^{\alpha(x,Tx)\alpha(y,Ty)} \leq \beta(M(x, y))M(x, y) + l, \forall\ x, y \in X \text{ and } l \geq 1. \quad (3.1)$$

Where $M(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Tx).d(Ty,y)}{d(x,y)}, \frac{d(x,Tx)(1+d(Ty,y))}{1+d(x,y)}\}$

Suppose that if T is continuous and

If there exists $x_0 \in X$ such that $(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof: Let $x_0 \in X$ such that $(x_0, Tx_0) \geq 1$. Construct a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$, $\forall\ n \in N$.

If $x_{n+1} = x_n$, for some $n \in N$, then $Tx_n = x_n$ and we are done.

So, we suppose that $(x_n, x_{n+1}) > 0$, $\forall\ n \in N$.

Since T is α −admissible, there exists $x_0 \in X$ such that $(x_0, Tx_0) \geq 1$ which implies $(x_0, x_1) \geq 1$.

Similarly, we can say that $(x_1, x_2) = \alpha(Tx_0, T^2x_0) \geq 1$.

By continuing this process, we get

$$(x_n, x_{n+1}) \geq 1, \forall\ n \in N. \quad (3.2)$$

By using equation (3.2), we have

$d(x_n, x_{n+1}) + l = d(Tx_{n-1}, Tx_n) + l \leq (d(Tx_{n-1}, Tx_n) + l)^{\alpha(x_{n-1},Tx_{n-1})\alpha(x_n,Tx_n)}$.

Now using equation (3.1), we get

$$d(x_n, x_{n+1}) + l \leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n) + l, \quad (3.3)$$

$(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), (x_{n-1}, Tx_{n-1}), (x_n, Tx_n), \frac{d(x_{n-1},Tx_{n-1}).d(Tx_n,x_n)}{d(x_{n-1},x_n)},$ $\frac{d(x_{n-1},Tx_{n-1})(1+d(Tx_n,x_n))}{1+d(x_{n-1},x_n)}\}$

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$$= \max \{(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

Assume that if possible $d(x_n, x_{n+1)} > d(x_{n-1}, x_n)$.

Then, $(x_{n-1}, x_n) = d(x_n, x_{n+1})$.Using

this in equation (3.3), we get

$$(x_n, x_{n+1}) < \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \tag{3.4}$$

$\Rightarrow (x_n, x_{n+1}) < d(x_n, x_{n+1})$, which is a contradiction.So

$(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \forall\, n.$

It follows that the sequence $\{(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of positive real numbers. So, it is convergent and suppose that $\lim_{n \to \infty} (x_n, x_{n+1}) = d$. Clearly, $d \geq 0$.

Claim: $d = 0$.

Equation (3.4) implies that

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq (d(x_{n-1}\ , x_n) \leq 1,$$

Which implies that $\lim_{n \to \infty} (d(x_{n-1}, x_n) = 1$.

Using the property of the function β, we conclude that

$$\lim_{n \to \infty} (x_n, x_{n+1}) = 0. \tag{3.5}$$

In the similar way, we can prove that

$$\lim_{n \to \infty} (x_n, x_{n+2}) = 0. \tag{3.6}$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\in > 0$ and sequences (k) and $n(k)$ such that for all positive integers k, we have

$n(k) > m(k) > k, d(x_{n(k)}, x_{m(k)}) \geq \in$ and $d(x_{n(k)}, x_{m(k)-1}) < \in$.

By the triangle inequality, we have

$$\in \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)+1}) + d(x_{m(k)-1}, x_{m(k)})$$

$$< \in + d(x_{m(k)-1}, x_{m(k)+1}) + d(x_{m(k)-1}, x_{m(k)}),$$

for all $k \in \mathbb{N}$.

Taking the limit as $k \to +\infty$ in the above inequality and using equations (3.5) and (3.6), we get

$$\lim_{k \to +\infty} (x_{n(k)}, x_{m(k)}) = \in. \tag{3.7}$$

Again, by triangle inequality, we have

$$d(x_{n(k)}, x_{m(k)}) - d(x_{m(k)-1}, x_{m(k)}) - d(x_{n(k)-1}, x_{n(k)}) \le d(x_{n(k)-1}, x_{m(k)-1})$$

$$d(x_{n(k)-1}, x_{m(k)-1}) \le d(x_{m(k)}, x_{m(k)-1}) + d(x_{n(k)}, x_{m(k)}) + d(x_{n(k)-1}, x_{n(k)}).$$

Taking the limit as $k \to +\infty$, together with (3.5) - (3.7), we deduce that

$$\lim_{k \to +\infty} (x_{n(k)-1}, x_{m(k)-1}) = \in. \tag{3.8}$$

From equations (3.1), (3.2), (3.6) and (3.8), we get

$$d(x_{n(k)}, x_{m(k)}) + l \le (d(x_{n(k)}, x_{m(k)}) + l)^{\alpha(x_{n(k)-1}, Tx_{n(k)-1})\alpha(x_{m(k)-1}, Tx_{m(k)-1})},$$

$$= (d(Tx_{n(k)-1}, Tx_{\ (\ )}) + l^{\ \alpha(x_{n(k)-1}, Tx_{n(k)-1})\alpha(x_{m(k)-1}, Tx_{m(k)-1})}$$

$$\le (M(x_{n(k)-1}, x_{m(k)-1})M(x_{n(k)-1}, x_{m(k)-1}) + l \tag{3.9}$$

$$M(x_{n(k)-1}, x_{m(k)-1}) = \max \{d(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), \frac{d(x_{n(k)-1}, Tx_{n(k)-1}).d(Tx_{m(k)-1}, x_{m(k)-1})}{d(x_{n(k)-1}, x_{m(k)-1})}, \frac{d(x_{n(k)-1}, Tx_{n(k)-1})(1+d(Tx_{m(k)-1}, x_{m(k)-1}))}{1+d(x_{n(k)-1}, x_{m(k)-1})}\},$$

$$= \max \{(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), \frac{d(x_{n(k)-1}, x_{n(k)}).d(x_{m(k)-1}, x_{m(k)})}{d(x_{n(k)-1}, x_{m(k)-1})}, \frac{d(x_{n(k)}, x_{n(k)-1})(1+d(x_{m(k)-1}, x_{m(k)}))}{1+d(x_{n(k)-1}, x_{m(k)-1})}\}.$$

Taking $k \to \infty$, we have

$(x_{n(k)-1}, x_{m(k)-1}) = \max \{\in, 0, 0, 0, 0\}$.So,

equation (3.9) implies that

$$d(x_{n(k)+1}, x_{m(k)+1}) \le \beta(M(x_{n(k)}, x_{m(k)})M(x_{n(k)}, x_{m(k)}) \le 1 ,$$

Letting $k \to \infty$, we get

$\lim_{k\to\infty} (d(x_{n(k)}, x_{m(k)}) = 1.$

By using definition of β function, we get

$\Rightarrow \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = 0 < \in$, which is a contradiction.

Hence, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete space, so $\{x_n\}$ is convergent and assume that $x_n \to x$ as $n \to \infty$.

Since T is continuous, then we have

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

So, x is a fixed point of T.

Theorem 3.2. Assume that all the hypothesis of Theorem 3.1 hold. Adding the following condition:

If $x = Tx$, then $(x, Tx) \geq 1$.

We obtain the uniqueness of fixed point.

Proof: Let z and z^* be two distinct fixed point of T in the setting of Theorem 3.1 and above defined condition holds, then

$(z, Tz) \geq 1$ and $\alpha(z^*, Tz^*) \geq 1$.

So, $d(Tz, Tz^*) + l \leq (d(Tz, Tz^*) + l)^{\alpha(z,Tz)\alpha(z^*,Tz^*)}$

$$\leq \beta(M(z, z^*))M(z, z^*) + l. \tag{3.10}$$

Where $M(z, z^*) = \max\{d(z, z^*), d(Tz, z), d(Tz^*, z), , \frac{d(z,Tz).d(T z^*,z^*)}{d(z,z^*)}, \frac{d(z,Tz)(1+d(T z^*,z^*))}{1+d(z,z^*)}\}$

$= d(z, z^*)$.

So, equation (3.10) implies

$$d(z, z^*) = d(Tz, Tz^*) \leq \beta(d(z, z^*))d(z, z^*)$$

$$\Rightarrow (d(z, z^*)) = 1$$

$$\Rightarrow (z, z^*) = 0 \Rightarrow z = z^*.$$

Corollary 3.3.(Farhan *et al.* [2]) Let (X, d) be a complete RMS and $T : X \rightarrow X$ be an α −admissible mapping. Assume that there exists a function $\beta: [0, \infty) \rightarrow [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and

$$(d(Tx, Ty) + l)^{\alpha(x,Tx)\alpha(y,Ty)} \leq \beta(d(x, y))d(x, y) + l$$

for all $x, y \in X$ where $l \geq 1$. Suppose that if T is continuous and there exists $x_0 \in X$ such that $(x_0, Tx_0) \geq 1$, then f has a fixed point.

Proof: Taking $(x, y) = d(x, y)$ in Theorem 3.1, one can get the proof.

Corollary 3.4. (Farhan *et al.*[2]) Assume that all the hypotheses of Corollary 3.3 hold. Adding the following condition:

(a) If $x = Tx$, then $(x, Tx) \geq 1$,

we obtain the uniqueness of the fixed point of T.

Proof: Taking $(x, y) = d(x, y)$ in Corollary 3.3.

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